JOURNAL OF COMPUTATIONAL PHYSICS 34, 94–107 (1980)

Initial Boundary Value Problems for the Method of Lines*

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Received November 28, 1979

This paper treats the stability of the initial boundary value problem for the method of lines applied to hyperbolic and parabolic partial differential equations in one space dimension. The theory treats the case of variable coefficients and allows for very general boundary conditions. Several examples are given which illustrate the theory. The theory is analogous to that developed by Gustafsson, Kreiss, and Sundström for finite-difference methods.

INTRODUCTION

The method of lines is a technique for the numerical solution of time-dependent partial differential equations in which one replaces the spatial differentiation by appropriate finite differences on a discrete set of grid points. In this way one obtains a system of ordinary differential equations with time as the independent variable and with the approximate values of the solution at the grid points as the dependent variables. This approximation of initial boundary value problems for partial differential equations by initial value problems for systems of ordinary differential equations is called the method of lines.

The resulting system is then solved numerically using a standard method for solving initial value problems for ordinary differential equations. An attractive feature of this method is that high-quality software has been developed to solve the stiff systems of differential equations which usually result from this method.

This paper discusses the stability of initial boundary value problems for the method of lines applied to linear hyperbolic and parabolic partial differential equations in one space dimension. Assuming that the problem for the partial differential equation is well posed, the stability question considered here is whether or not the approximation by the method of lines is also well posed, i.e., stable. No attempt will be made to discuss the accuracy or efficiency of the method as a general computational technique.

The results obtained here are analogous to those obtained for finite-difference equations by Gustafsson, Kreiss, and Sundström [2], Varah [4], and for hyperbolic partial differential equations by Kreiss [3]. The paper by Gustafsson, Kreiss, and Sundström will hereafter be referred to as GKS.

* This report was prepared as a result of work performed under NASA Contract No. NAS1-14101 at ICASE, NASA Langley Research Center, Hampton, Va. 23665.

0021-9991/80/010094–14$02.00/0

The main theorem of the paper concerning stability is stated in Section 1 for linear hyperbolic systems and its application is illustrated by several examples. The proof of the theorem and its extension to parabolic systems is sketched in Section 2. Section 3 treats variable coefficients and the case of two boundaries. Finally, Section 4 clarifies the relationship of these results to those of Gary [1].

1. The Hyperbolic Initial Boundary Value Problem

Consider the hyperbolic system of partial differential equations

$$u_t = Au_x + F(t, x) \tag{1.1}$$

on the half-line $x \geqslant 0$ with $t \geqslant 0$. At the boundary $x = 0$, we have the boundary conditions

$$Tu(t, 0) = g(t) \tag{1.2}$$

and at $t = 0$ we have, for simplicity, the initial condition

$$u(0, x) = 0. \tag{1.3}$$

The variable u is an n-vector, $u = (u_1, ..., u_n)'$, and since the system (1.1) is hyperbolic, the matrix A is diagonalizable with real eigenvalues. We also assume that A is non-singular, and that the rank of T is equal to the number of negative eigenvalue of A as is required for well-posedness.

We will assume that the above initial boundary value problem is well posed. The theorem for well-posedness of hyperbolic initial boundary value problems of this type is the following (cf. Kreiss [3]).

THEOREM 1. *The initial boundary value problem* (1.1)–(1.3) *is well posed if and only if it has no eigensolutions.*

Here eigensolutions are defined as follows:

DEFINITION. An eigensolution for the system (1.1)–(1.3) is a function $v(x, s)$ satisfying the following:

(a) $sv = Av_x$ on $x \geqslant 0$,

(b) $\mathscr{Re}\, s \geqslant 0$,

(c) for $\mathscr{Re}\, s > 0$, $v(x, s)$ is bounded as $x \to \infty$,

(d) for $\mathscr{Re}\, s = 0$, $v(x, s) = \lim_{\epsilon \to 0+} v(x, s + \epsilon)$, where $v(x, s + \epsilon)$ satisfies (c) and (a) (with s replaced by $s + \epsilon$), and

(e) $Tv(0, s) = 0$.

To approximate the initial boundary value problem (1.1)–(1.3) by the method of lines, a set of grid points $\{x_i\}_{i=0}^{\infty}$, $x_i = ih$, is introduced and the differential equation (1.1) is replaced by

$$\frac{dv^i}{dt} = ADv^i + f(t, x_i) \qquad \text{for} \quad i \geqslant r. \tag{1.5}$$

The new variables $v^i(t)$ approximate $u(t, x_i)$ and the difference operator D is defined by

$$Dv^m = \sum_{j=-r}^{r} d_j v^{m+j}, \tag{1.6}$$

such that Eq. (1.5) is a consistent approximation. The coefficients d_j may be matrices as would be the case if different variables were differenced by different schemes. Equation (1.5) is the approximation of Eq. (1.1) on the interior of the grid; for the boundary region where Eq. (1.5) cannot be applied we use different difference schemes at each point. We have

$$\frac{dv^k}{dt} = AD_k v^k + f(t, x_k), \qquad 0 < k < r, \tag{1.7}$$

where the difference operators D_k are given by

$$D_k v^k = \sum_{j=-k}^{2r-k} d_{kj} v^{k+j}, \qquad 0 \leqslant k < r. \tag{1.8}$$

On the boundary itself we have the boundary condition (1.2) and possibly some interpolation. This can be written

$$\sum_{j=0}^{r} T_j v^j = \tilde{g}(t). \tag{1.9}$$

In order to completely determine the variable v^0 additional conditions obtained from the differential equation may also be applied. This can be written

$$S \frac{dv^0}{dt} = AD_0 v^0 + f(t, 0), \tag{1.10}$$

where D_0 is given by Eq. (1.8) for $k = 0$. We will assume that the rows of T_0 and S form a linearly independent set of n vectors, that is, all components of the vector v^0 can be determined by Eqs. (1.9) and (1.10).

Finally, the initial conditions are taken to be

$$v^i = 0, \qquad i \geqslant 0. \tag{1.11}$$

We will, of course, need that the Cauchy problem for Eq. (1.5) is stable. For convenience we will make the following stronger assumption which was also made in GKS.

Assumption 1. The difference equation (1.5) is either totally dissipative or totally nondissipative, that is, the equation

$$\det \left| sI - A \sum_{j=-r}^{r} d_j e^{ij\theta} \right| = 0$$

satisfies either

$$\mathscr{R}e\, s \leqslant -\delta\theta^2 \qquad \text{for } |\theta| < \pi$$

for some positive constant δ, or

$$\mathscr{R}e\, s = 0 \qquad \text{for all } \theta.$$

The purpose of this paper is to determine the conditions under which the initial boundary value problem given by Eqs. (1.5)–(1.11) is stable. The conditions are best stated in terms of eigensolutions which are defined as follows.

Consider the difference equation given by the resolvent equation corresponding to Eq. (1.5),

$$sv^i = ADv^i = \sum_{j=-r}^{r} d_j v^{i+j}. \tag{1.12}$$

Let $\{v^i(s)\}_{i=0}^{\infty}$ be a solution to Eq. (1.12) for $\mathscr{R}e\, s > 0$ such that

$$v^i(s) \to 0 \qquad \text{as} \qquad i \to +\infty. \tag{1.13}$$

If $\{v^i(s)\}$ is such a solution which in addition satisfies the homogeneous boundary conditions

$$sv^k = AD_k v^k, \qquad 0 < k < r, \tag{1.14}$$

$$sSv^0 = AD_0 v^0, \tag{1.15}$$

and

$$\sum_{j=0}^{r} T_j v^j = 0, \tag{1.16}$$

then $\{v^i(s)\}$ is an eigensolution.

Eigensolutions are also defined when the real part of s is zero; in this case Eq. (1.13) must be replaced by the condition

$$v^i(s_0) = \lim_{\epsilon \to 0^+} v^i(s_0 + \epsilon), \tag{1.17}$$

where $\{v^i(s_0 + \epsilon)\}$ satisfies both Eq. (1.12) for $s = s_0 + \epsilon$ and condition (1.13).

The main result can now be stated.

MAIN THEOREM. *A necessary and sufficient condition for the method-of-lines initial boundary value problem* (1.5)–(1.11) *to be stable is that there exist no eigensolution to the problem.*

The proof will be deferred until the next section in order to give immediate examples of its application. The first two examples given are very simple and serve to illustrate the theory without algebraic encumbrances. The last two examples are more typical of boundary conditions encountered in applications and require more algebraic manipulation.

EXAMPLE 1. Consider the partial differential equation

$$u_t = u_x + f \tag{E1.1}$$

on the region $x \geqslant 0$, $t \geqslant 0$. No boundary condition is needed at $x = 0$. Consider the method-of-lines approximation

$$\frac{dv^i}{dt} = \frac{v^{i+1} - v^i}{h} + f(t, x_i), \qquad i = 0, 1, \ldots, \tag{E1.2}$$

and (E1.2) is also the boundary condition at $i = 0$.

To check for eigensolutions we solve

$$hsv^i = v^{i+1} - v^i, \tag{E1.3}$$

which has the solution

$$v^i = (1 + sh)^i v^0.$$

For $\mathscr{Re}\, s > 0$ there are no solutions to (E1.3) with $v^i \to 0$ as $i \to \infty$. Therefore, since (E1.2) has no eigensolutions, it is stable.

EXAMPLE 2. Consider the same partial differential equation as in Example 1, but with the method-of-lines approximation

$$\frac{dv^i}{dt} = \frac{(v^{i+1} - v^{i-1})}{2h}, \qquad i \geqslant 1. \tag{E2.1}$$

This requires a boundary condition at $i = 0$ since (E2.1) cannot be applied there. We take

$$\frac{dv^0}{dt} = \frac{(v^2 - v^0)}{2h}. \tag{E2.2}$$

The resolvent equation is

$$2shv^i = v^{i+1} - v^{i-1}. \tag{E2.3}$$

To solve this equation set $v^i = \kappa^i v^0$ and substitute in Eq. (E2.3) then solve

$$2sh\kappa = \kappa^2 - 1.$$

The solution for $|\kappa| \leqslant 1$ is

$$\kappa = sh - [(sh)^2 + 1]^{1/2}.$$

Substituting this in the boundary condition we have

$$2sh = (sh - [(sh)^2 + 1]^{1/2})^2 - 1$$

which has the solution $s = 0$. Thus a solution to the resolvent equation and the boundary conditions exists, but since the real part of s is zero, condition (1.17) must be satisfied. Setting $sh = \epsilon$ we have

$$|\kappa| = |\epsilon - (1 + \epsilon^2)^{1/2}| < 1$$

so that $\{v^i\}$ satisfies condition (1.17), and therefore is an eigensolution. This shows that this second example is not stable.

The next two examples are taken from scheme D in the paper by Gary [1] and will be discussed again in section 4.

EXAMPLE 3. The partial differential equation is

$$u_t = -u_x \qquad \text{on} \quad x \geqslant 0, \qquad t \geqslant 0; \tag{E3.1}$$

the boundary condition required at $x = 0$ is

$$u(t, x = 0) = g(t). \tag{E3.2}$$

For the method-of-lines approximation take

$$\frac{dv^j}{dt} = \frac{(-v^{j-2} + 8v^{j-1} - 8v^{j+1} + v^{j+2})}{12h}, \qquad j \geqslant 2. \tag{E3.3}$$

For the boundary conditions take

$$\frac{dv^1}{dt} = \frac{(3v^0 + 10v^1 - 18v^2 + 6v^3 - v^4)}{12h} \tag{E3.4}$$

and

$$v^0 = g(t). \tag{E3.5}$$

We now examine this problem for eigensolutions. The resolvent equation for (E3.3) is

$$12hsv^j = -v^{j-2} + 8v^{j-1} - 8v^{j+1} + v^{j+2}. \tag{E3.6}$$

To solve this difference equation set $v^j = \kappa^j v^0$, obtaining a polynomial equation for κ,

$$12hs = (-1 + 8\kappa - 8\kappa^3 + \kappa^4)\,\kappa^{-2} = k(\kappa)\,\kappa^{-2}. \tag{E3.7}$$

For $\mathscr{Re}\, s$ large there are two roots of (E3.7) inside the unit circle and $\mathscr{Re}\, s$ is zero for κ on the unit circle. Therefore, in general there are two roots of (E3.7) with modulus less than unity when $\mathscr{Re}\, s > 0$. Denote these two roots as $\kappa_1(s)$ and $\kappa_2(s)$. If $\{v^j\}$ is to be an eigensolution then it must have the form

$$v^j = \alpha_1(\kappa_1(s))^j + \alpha_2(\kappa_2(s))^j.$$

The homogeneous boundary condition at $j = 0$ gives $\alpha_2 = -\alpha_1$; we may take $\alpha_1 = 1$ by homogeneity. The boundary condition at $j = 1$ gives

$$\begin{aligned}12hs(\kappa_1 - \kappa_2) &= 10(\kappa_1 - \kappa_2) - 18(\kappa_1^2 - \kappa_2^2) + 6(\kappa_1^3 - \kappa_2^3) - (\kappa_1^4 - \kappa_2^4),\\ 12hs &= 10 - 18(\kappa_1 + \kappa_2) + 6(\kappa_1^2 + \kappa_1\kappa_2 + \kappa_2^2) - (\kappa_1^2 + \kappa_2^2)(\kappa_1 + \kappa_2).\end{aligned} \tag{E3.8}$$

A second expression for s is obtained from (E3.7). Since

$$12hs = k(\kappa_1)\,\kappa_1^{-2} = k(\kappa_2)\,\kappa_2^{-2},$$

then

$$\begin{aligned}12hs &= (k(\kappa_1) - k(\kappa_2))/(\kappa_1^2 - \kappa_2^2)\\ &= \kappa_1^2 + \kappa_2^2 - 8(\kappa_1^2 + \kappa_1\kappa_2 + \kappa_2^2)/(\kappa_1 + \kappa_2) + 8/(\kappa_1 + \kappa_2).\end{aligned} \tag{E3.9}$$

Equating these two expressions for s and setting

$$\kappa_1 + \kappa_2 = y, \qquad \kappa_1\kappa_2 = x$$

gives

$$y^3 - 5y^2 + 10y - 10 - 2xy + 4x + 8(x + 1)/y = 0. \tag{E3.10}$$

A second equation for x and y obtained from (E3.7) is

$$(k(\kappa_1)\,\kappa_1^{-2} - k(\kappa_2)\,\kappa_2^{-2})/(\kappa_1 - \kappa_2) = 0$$

which simplifies to

$$y = 8x(x + 1)/(x^2 + 1). \tag{E3.11}$$

Substituting (E3.11) in (E3.10) we obtain

$$-11x^8 + 246x^7 + 960x^6 + 994x^5 + 34x^4 - 286x^3 + 88x^2 - 10x + 1 = 0. \quad \text{(E3.12)}$$

Solving for the roots of (E3.12) numerically and using (E3.11) to obtain y and thus κ_1 and κ_2, and then using (E3.7) to obtain s, we see that there are no eigensolutions and thus the problem is stable. Note that by diving by $(\kappa_1 - \kappa_2)$ in obtaining (E3.11) we have also treated the case of $\kappa_1 = \kappa_2$.

EXAMPLE 4. The partial differential equation is

$$u_t = u_x \qquad \text{on} \quad x \geqslant 0, t \geqslant 0, \tag{E4.1}$$

and no boundary condition is required at $x = 0$.

For the method-of-lines approximation take

$$\frac{dv^j}{dt} = \frac{(v^{j-2} - 8v^{j-1} + 8v^{j+1} - v^{j+2})}{12h}, \qquad j \geqslant 2. \tag{E4.2}$$

For the boundary conditions, take

$$\frac{dv^1}{dt} = -\frac{(3v^0 + 10v^1 - 18v^2 + 6v^3 - v^4)}{12h} \tag{E4.3}$$

and

$$\frac{dv^0}{dt} = -\frac{(25v^0 - 48v^1 + 36v^2 - 16v^3 + 3v^4)}{12h}. \tag{E4.4}$$

We now check for eigensolutions to this problem. As in Example 3, we solve for

$$v^j = \alpha_1 \kappa_1{}^j + \alpha_2 \kappa_2{}^j,$$

where κ_1 and κ_2 are the roots of

$$12hs = (1 - 8\kappa + 8\kappa^3 - \kappa^4)\,\kappa^{-2} \tag{E4.5}$$

satisfying $|\kappa| \leqslant 1$.

The boundary conditions yield two equations for α_1 and α_2. Let $\sigma = 12hs$, then we obtain

$$\begin{aligned} \sigma(\alpha_1 + \alpha_2) &= \alpha_1 g_0(\kappa_1) + \alpha_2 g_0(\kappa_2), \\ \sigma(\alpha_1\kappa_1 + \alpha_2\kappa_2) &= \alpha_1 g_1(\kappa_1) + \alpha_2 g_1(\kappa_2), \end{aligned} \tag{E4.6}$$

where

$$\begin{aligned} g_0(\kappa) &= -(25 - 48\kappa + 36\kappa^2 - 16\kappa^3 + 3\kappa^4), \\ g_1(\kappa) &= -(3 + 10\kappa - 18\kappa^2 + 6\kappa^3 - \kappa^4). \end{aligned}$$

If there exists a nontrivial solution to (E4.6), then the problem is unstable. A nontrivial solution to (E4.6) requires that

$$0 = \frac{1}{\kappa_1 - \kappa_2} \det \begin{vmatrix} g_0(\kappa_1) - \sigma & g_0(\kappa_2) - \sigma \\ g_1(\kappa_1) - \sigma\kappa_1 & g_1(\kappa_2) - \sigma\kappa_2 \end{vmatrix}. \tag{E4.7}$$

If κ satisfies (E4.5), then

$$g_0(\kappa) - \sigma = \kappa^{-2}(1 - \kappa)^5 (3\kappa - 1)$$

and

$$g_1(\kappa) - \sigma\kappa = -\kappa^{-1}(1 - \kappa)^5,$$

so that Eq. (E4.7) reduces to

$$-(1 - \kappa_1)^5 (1 - \kappa_2)^5 \kappa_1^{-2}\kappa_2^{-2} = 0.$$

Therefore an eigensolution exists only if κ_1 or κ_2 equals 1. If one root, say κ_1, is equal to 1, then $s = 0$, and from (E4.5)

$$\kappa_2 = 4 - (15)^{1/2} \qquad \text{or} \qquad \kappa_2 = -1.$$

To see if $\kappa_1 = 1$ gives rise to an eigensolution we consider how κ_1 depends on s when s is perturbed from 0 to ϵ. Set $\kappa = 1 + \delta$ in (E4.5), then

$$12hs = 12\delta + O(\delta^2).$$

So for $\mathscr{Re}\, s > 0$, then $|\kappa| > 1$, so $s = 0$, $\kappa = 1$ does not give an eigensolution. Again, one can check that the possibility of $\kappa_1 = \kappa_2$ has been covered by dividing (E4.7) by $(\kappa_1 - \kappa_2)$. We conclude there are no eigensolutions and this example is stable.

2. The Proof of the Main Theorem

The proof of stability for initial boundary value problems for the method of lines is essentially the same as the analogous theorem for difference equations given in GKS. Only a sketch of the proof will be given.

The first step is to take the Laplace transform with respect to t of Eq. (1.5). Let s be the dual variable. We obtain

$$s\hat{v}^j = AD\hat{v}^j + \hat{f}(s, x_j) \tag{2.1}$$

and similar equations for the boundary conditions. Next rewrite the equation as a one-step difference equation in the x-direction, i.e.,

$$w^{\nu+1} = M(s)\, w^{\nu} + G^{\nu}, \qquad \nu \geqslant 0, \tag{2.2}$$

where

$$w^{\nu} = (\hat{v}^{2r+\nu-1}, \ldots, \hat{v}^{\nu})'.$$

If we assume for simplicity that the matrix d_r in Eq. (1.6) is nonsingular, then $M(s)$ has the form

$$M(s) = -\begin{pmatrix} d_r^{-1}d_{r-1} & \cdots & d_r^{-1}(d_0 - A^{-1}s) & \cdots & d_r^{-1}d_{-r} \\ -I & & & & \\ 0 & & -I & & 0 \end{pmatrix}$$

and

$$G^{\nu} = (d_r^{-1}\hat{f}_{\nu-r}, 0, \ldots, 0)'.$$

We also write the boundary conditions in terms of w^0, obtaining

$$\tilde{T} w^0 = \tilde{g}. \tag{2.3}$$

The matrix $M(s)$ is like the matrix $M(z)$ occurring in GKS, the main difference being that $M(z)$ depends on z for $|z| \geqslant 1$ while $M(s)$ depends on s for $\mathscr{Re}\, s \geqslant 0$. The relationship between z and s can be thought of as $z = \exp(hs)$, although this is not rigorous.

Before actually proving the theorem it is necessary to transform $M(s)$ to a normal form. The normal form is the same as that of Theorem 9.1 of GKS. This normal form transforms $M(s)$ into block form with each block containing different types of eigenvalues.

With $M(s)$ in this normal form it is possible to construct a Hermitian matrix $H(s)$ satisfying

$$M^*HM - H \geqslant \delta_1 h\, \mathscr{Re}\, s. \tag{2.4}$$

(M^* denotes the conjugate transpose of M.) If there are no eigensolutions, $H(s)$ can be constructed so that

$$H + \tilde{T}^*\tilde{T} \geqslant \delta_2, \tag{2.5}$$

where δ_1 and δ_2 are positive constants and H is bounded in norm independently of s.

Define a norm by

$$\|f\|_+^2 = \sum_{\nu=0}^{\infty} |f^{\nu}|^2.$$

With these preliminaries, the proof may now be established.

By inequality (2.5)

$$
\begin{aligned}
h\eta\delta_1 \| w \|_+^2 &\leqslant \sum_{\nu=0}^{\infty} (Mw^\nu, HMw^\nu) - (w^\nu, Hw^\nu) \\
&= \sum_{\nu=0}^{\infty} (w^{\nu+1}, Hw^{\nu+1}) - (w^\nu, Hw^\nu) \\
&\quad - 2\,\mathscr{Re} \sum_{\nu=0}^{\infty} (G^\nu, Hw^{\nu+1}) + \sum_{\nu=0}^{\infty} (G^\nu, HG^\nu) \\
&\leqslant -(w^0, Hw^0) + h \| w \|_+^2 + C(h^{-1} + 1) \| G \|_+^2 \\
&\leqslant -\delta_2 | w^0 |^2 + | \tilde{T}w^0 |^2 + h \| w \|_+^2 + C(h^{-1} + 1) \| G \|_+^2 .
\end{aligned}
$$

Rewriting the above in terms of $\hat{v}$ and defining a norm for the boundary

$$
| f |_b^2 = \sum_{j=0}^{2r-1} | f^j |^2,
$$

we obtain

$$
h(\eta\delta_1 - 1) \| \hat{v} \|_+^2 + \delta_2 | \hat{v} |_b^2 \leqslant C(| \hat{g}(s) |_b^2 + h \| \hat{f} \|_+^2).
$$

All that remains is to use Parseval's equality to write the above inequality in tems of the original, untransformed variables. This is done rather concisely if we introduce the norms

$$
\| f \|_\eta^2 = \int_0^\infty e^{-2\eta t} \| f \|_+^2 \, h \, dt
$$

and

$$
| f |_\eta^2 = \int_0^\infty e^{-2\eta t} | f |_b^2 \, dt,
$$

and use Parseval's relation

$$
\int_{\eta - i\infty}^{\eta + i\infty} | f(\eta + i\tau) |^2 \, d\tau = \int_0^\infty e^{-2\eta t} | f(t) |^2 \, dt.
$$

Then the above inequality becomes, for η sufficiently large,

$$
\eta \| v \|_\eta^2 + | v |_\eta^2 \leqslant C(\| f \|_\eta^2 + | g |_\eta^2) \tag{2.6}
$$

for some constant C independent of η. This inequality defines the meaning of stability for the hyperbolic initial boundary value problem given by Eqs. (1.5)–(1.11). This completes the proof of the Main Theorem.

The theory for parabolic initial boundary value problems is essentially the same as that for hyperbolic problems. The main difference is that in the parabolic case the matrix $H(s)$ does not satisfy inequality (2.4) but rather

$$M^*HM - H \geqslant \delta_1 h \,\mathscr{Re}\, s^{1/2}$$

so that inequality (2.6) becomes

$$\eta^{1/2} \| v \|_\eta^2 + | v |_\eta^2 \leqslant C(\| f \|_\eta^2 + | g |_\eta^2).$$

3. Variable Coefficients and Two Boundaries

In actual practice the problem (1.1) would be solved, not on a half-line $x \geqslant 0$, but on an interval such as $0 \leqslant x \leqslant 1$. Also, the coefficient matrices might be functions of t and x, and in addition, lower-order terms could be present in Eq. (1.1). All of these considerations can be handled following the treatment given in GKS.

Stability, as used in this paper, is preserved under the effects of both lower-order terms and variable coefficients, provided they are smooth enough. That is, one need only examine the "frozen coefficient problems" without lower-order terms to determine stability. The frozen coefficient problems are those constant coefficient initial boundary value problems obtained by fixing all the coefficients at the values they take on for each value of t and x. The stability of the initial boundary value problem with two boundaries can be shown to follow from the stability of each of the corresponding half-plane problems which are those obtained by freezing the coefficients at their values on the boundary, ignoring the effect of the opposite boundary, and extending the domain to infinity. If each of these half-plane problems is stable, then so is the original problem.

4. A Discussion of Examples 3 and 4

This section discusses the application of the preceeding results to scheme D of the paper by Gary [1]. This scheme is a method-of-lines approximation to the problem

$$\begin{aligned} &u_t + u_x = 0, \qquad 0 \leqslant x \leqslant 0.5, \\ &u(t, 0) = -\sin 2\pi t, \\ &u(0, x) = \sin 2\pi x. \end{aligned} \tag{4.1}$$

The approximation is

$$\frac{dv^j}{dt} = -\frac{(v^{j-2} - 8v^{j-1} + 8v^{j+1} - v^{j+2})}{12h}, \qquad 2 \leqslant j \leqslant J - 2 \tag{4.2}$$

with the boundary conditions

$$
\begin{aligned}
v^0(t) &= -\sin 2\pi t, \\
\frac{dv^1}{dt} &= -\frac{(3v^0 + 10v^1 - 18v^2 + 6v^3 - v^4)}{12h}, \\
\frac{dv^{J-1}}{dt} &= -\frac{(-v^{J-4} + 6v^{J-3} - 18v^{J-2} + 10v^{J-1} + 3v^J)}{12h}, \\
\frac{dv^J}{dt} &= -\frac{(3v^{J-4} - 16v^{J-3} + 36v^{J-2} - 48v^{J-1} + 25v^J)}{12h},
\end{aligned}
\tag{4.3}
$$

and

$$v^j(0) = \sin 2\pi hj, \qquad j = 0,..., J.$$

Example 3 treated the half-plane problem arising from the first two boundary conditions, i.e., those at $x = 0$, while Example 4, treated the half-plane problem for the last two boundary conditions, i.e., those for $x = 0.5$. These two examples and the theory outlined in Section 3 show that the initial boundary value problem (4.2)–(4.3) is stable in the sense of this paper.

However, under Gary's definition this scheme is unstable. His definition of stability does not allow for any growing modes in the solution of the approximation, i.e., all eigenvalues of the system of differential equations (4.3) must have a nonpositive real part. The definition of stability employed in this paper however does allow for eigenvalues with positive real part. Gary's definition of stability, which is never explicitly stated, is stronger than that of this paper in that it restricts the exponential growth of the solution of the approximation to be no more than that of the solution of the differential equation. It is this author's opinion that such a restriction is in reality a restriction on the accuracy of the approximation and therefore should not be considered in questions of stability.

Under Gary's definition stability may be influenced by the number of grid points, J, employed. Thus, under his definition, scheme (4.3) is stable for $J = 5$, while for $J = 10$ or $J = 20$ it is unstable. Under the definition used in this paper, scheme (4.3) is stable for all values of J, but because of the eigenvalues with positive real part the scheme for $J = 10$ will be less accurate than that for $J = 5$ over long time integrations.

In summary, the advantages of the definition of stability used in this paper are that the stability is independent of the number of grid points, the theory encompasses variable coefficients and lower-order terms, and there exists a procedure for deciding the stability of any given initial boundary value problem.

Acknowledgments

I especially thank Dr. David Gottlieb and Dr. Joseph Oliger for many stimulating conversations regarding the content of this paper.

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